WAVES OF ENVELOPES IN ELECTRON STREAMS

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We study stationary waves of envelopes in a nonrelativistic electron stream on a fixed ion background. It is shown that the velocity of the stationary wave of an envelope is always equal to the velocity of the unperturbed electron stream. In the adiabatic approximation it is found that each perturbation of the envelopes propagates with velocity of the unperturbed stream. The result is of interest for the theory of nonlinear waves in dispersive media.

1. In recent years there have been investigations dealing with the fact that for nonlinear wave processes in dispersive media the most characteristic effects evidently are various effects of nonlinear self-stress [1-7]. The latter include, for example, the self-focusing of stationary-wave beams [2]. In the nonstationary case, effects of self-action appear as a nonlinear space–time deformation of amplitude and phase of the envelopes of a wave packet [3, 4, 6, 7]. This allows us to treat nonstationary effects of self-action as a propagation into the medium of peculiar waves of the envelopes. In a number of cases the envelope waves undergo accumulated distortion, and for certain conditions Reimann envelope waves are possible. In a medium with a relaxation nonlinearity, shock waves of the envelopes can exist [7].

For a consideration of the effects of self-action in nonlinear optics it is usual to assume the presence of a cubic nonlinearity in the medium. Of course this does not exhaust all the possibilities in nonlinear media, in which self-action effects are possible. In particular, we should take note of dispersive media with a hydrodynamic nonlinearity $(\mathbf{v}\,\nabla)\,\mathbf{v}$, where $\mathbf{v}$ is the velocity and ∇ is the dell operator [8, 9]. The quasishock waves considered in [8] are essentially shock waves of the envelopes. In [9], it is shown that waves of finite amplitude of ionic sound in a plasma experience self-action, leading to the decay of the homogeneous wave front. This effect is similar to self-focusing in nonlinear optics [2, 5]. All of this indicates the advisability of studying wave processes in dispersive media with a hydrodynamic nonlinearity in terms of the nonlinear self-action.

The aim of the present work is to attempt to construct a theory in which wave processes in electron streams can be discussed as a space–time deformation of the envelopes.

2. The principal results of the linear theory of homogeneous wave processes in a nonrelativistic electron stream on a fixed ion background reduces to the following. Each plane perturbation of the stream $f(x, t)$ can be represented in the form

$$f(x,t)=\int_{-\infty}^{+\infty} f_{\omega_1} e^{i(k_1 x-\omega_1 t)}\,d\omega_1+\int_{-\infty}^{+\infty} f_{\omega_2} e^{i(k_2 x-\omega_2 t)}\,d\omega_2 \tag{2.1}$$

$$k_1=k_1(\omega_1)=\frac{\omega_1-\omega_p}{v_0} \tag{2.2}$$

$$k_2=k_2(\omega_2)=\frac{\omega_2+\omega_p}{v_0} \tag{2.3}$$

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Here, v_0 is the velocity of the unperturbed electron stream, ω_p is the plasma frequency, and Eqs. (2.2) and (2.3) determine the fast and slow waves of the space charge, respectively. Transforming (2.1), we obtain

$$f(x, t) = \cos \frac{\omega_p}{v_0} x \cdot f\left(0, t - \frac{x}{v_0}\right)$$

i.e.,

$$f(x + l, t + \tau) = f(x, t) \quad (\tau = 2\pi/\omega_p, \quad l = v_0\tau)$$

The latter indicates that, moving with velocity of the unperturbed stream, the perturbation repeats itself in time τ. It is clear that the propagation of such a perturbation can conveniently be described following the space–time deformation of its envelope. Similarly to what has been done previously, we have

$$f(x, t) = \cos \omega_p t \cdot f(x - v_0 t, 0)$$

Thus, in the linear theory the envelope of a wave packet moves, without being deformed, with velocity of the unperturbed stream. This result follows from the law of dispersion of the waves of the space charge. On the other hand it attests to the advantage and visualizability of the representation of envelope waves in the electron stream.

3. For an investigation of nonlinear wave processes in an electron stream on a fixed ion background we initially have the following system of equations:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e}{m}\frac{\partial \varphi}{\partial x}, \quad \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \quad \frac{\partial^2 \varphi}{\partial x^2} = 4\pi e(n - n_0) \tag{3.1}$$

where v is the velocity of the electron stream, φ is the electrostatic potential, n is the electron density, n_0 is the ion density, e is the absolute value of the electron charge, and m is the electron mass.

In (3.1), we make the substitution of variables [9, 10]

$$x_1 = (x - u_1 t)\, k^*, \quad x_2 = (x - u_2 t)\, k^*$$
$$k^* = \frac{\omega_p}{|v_0 - u_1|}, \quad \omega_p^2 = \frac{4\pi e^2 n_0}{m}$$

where u_1 and u_2 are constants, and v_0 is the velocity of the unperturbed stream.

The original system becomes the following system of equations:

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left[\frac{\partial}{\partial x_1}V^2 + \frac{\partial}{\partial x^2}(V + u)^2\right] = 2\left(\frac{I}{V} - 1\right)$$
$$\frac{\partial I}{\partial x_1} + \frac{\partial I}{\partial x_2} = -u\frac{\partial}{\partial x_2}\left(\frac{I}{V}\right) \tag{3.2}$$

Here, we have introduced the notation

$$V = \frac{v - u_1}{v_0 - u_1}, \quad I = \frac{n}{n_0}V, \quad u = \frac{u_1 - u_2}{v_0 - u_1}$$

4. Let $\partial/\partial x_2 = 0$. This indicates that we seek solutions of (3.2) in the form of plane stationary waves

$$d^2\Phi / dx_1^2 = 2(1/\sqrt{\Phi} - 1) \tag{4.1}$$

where $\Phi = V^2$. We note that sgn $(v - u_1)$ = sgn $(v_0 - u_1)$, so that $V > 0$.

Integrating, we obtain

$$(d\Phi / dx_1)^2 = 4\{A - (\sqrt{\Phi} - 1)^2\} = 4F(A, \Phi) \quad (F(A, \Phi) = A - (\sqrt{\Phi} - 1)^2) \tag{4.2}$$

Hence

$$x_1 + \theta = \pm\int\frac{d\Phi}{2\sqrt{F(A, \Phi)}} \tag{4.3}$$

Here, A and θ are constants of integration. The quantity A corresponds to the amplitude of the wave, θ corresponds to its phase. For the condition

$$0 < A < 1$$

Eq. (4.3) determines the periodic function

$$\Phi = \Phi(A, x_1 + \theta) \tag{4.4}$$

Its period equals 2π.

5. We will seek solutions of the system (3.2) in the form of a longwave perturbation of the stationary wave moving with velocity u_1. Using Eq. (4.4) for this, instead of the two functions $V(x_1, x_2)$ and $I(x_1, x_2)$, we introduce into the investigation three functions: $A(x_1, x_2)$, $\theta(x_1, x_2)$, and $I(x_1, x_2)$. Imposing on A and θ the additional conditions

$$\frac{d\Phi}{dx_1} + \frac{d}{dx_2}(u + \sqrt{\Phi})^2 = \frac{\partial\Phi}{\partial x_1}$$

we obtain a system of equations, equivalent to (3.2):

$$\left(\frac{\partial A}{\partial x_1}\frac{\partial}{\partial A} + \frac{\partial\theta}{\partial x_1}\frac{\partial}{\partial x_1}\right)\Phi = -\frac{d\Phi}{dx_2} - 2n\frac{d}{dx_2}\sqrt{\Phi} \tag{5.1}$$

$$\left(\frac{\partial A}{\partial x_1}\frac{\partial}{\partial A} + \frac{\partial\theta}{\partial x_1}\frac{\partial}{\partial x_1}\right)\frac{\partial\Phi}{\partial x_1} = 2/I(\sqrt{\Phi} - 1) - \frac{\partial^2\Phi}{\partial x_1^2} - \frac{d}{dx_2}\left(\frac{\partial\Phi}{\partial x_1}\right) \tag{5.2}$$

$$\frac{\partial I}{\partial x_1} = u\frac{d}{dx_2}\left(\frac{I}{\sqrt{\Phi}}\right) - \frac{dI}{dx_2} \tag{5.3}$$

From (4.1) and (4.2), we obtain

$$\frac{\partial^2\Phi}{\partial x_1^2}\frac{\partial\Phi}{\partial A} - \frac{\partial\Phi}{\partial x_1}\frac{\partial}{\partial A}\left(\frac{\partial\Phi}{\partial x_1}\right) = -2, \quad \frac{\partial^2\Phi}{\partial x_1^2}\frac{d\Phi}{dx_2} - \frac{\partial\Phi}{\partial x_1}\frac{d}{dx_2}\left(\frac{\partial\Phi}{\partial x_1}\right) = -2\frac{dA}{dx_2}$$

Therefore, having solved the system (5.1), (5.2), for $\partial A/\partial x_1$ and $\partial\theta/\partial x_1$, we have

$$2\frac{\partial A}{\partial x_1} = -2\frac{dA}{dx_2} + 4u\frac{d\sqrt{\Phi}}{dx_2}\left(\frac{1}{\sqrt{\Phi}} - 1\right) + 2\Phi\frac{I-1}{\sqrt{\Phi}} \tag{5.4}$$

$$2\frac{\partial\theta}{\partial x_1} = -\frac{\partial^2\Phi}{\partial A\partial x_1}\left(\frac{d\Phi}{dx_2} + 2u\frac{d\sqrt{\Phi}}{dx_2}\right) - \frac{\partial\Phi}{\partial A}\left[2\frac{I-1}{\sqrt{\Phi}} - \frac{d}{dx_2}\left(\frac{\partial\Phi}{\partial x_1}\right)\right] \tag{5.5}$$

6. In order to calculate solutions of the system (5.3)-(5.5), we use the method of averaging [11]. In order to do this, we represent the sought functions in the form of the series

$$\begin{aligned} A(x_1, x_2) &= A_0(x_2) + \sum_{k=1}^{\infty} A_k(x_1, x_2) \\ \theta(x_1, x_2) &= \theta_0(x_2) + \sum_{k=1}^{\infty} \theta_k(x_1, x_2) \\ I(x_1, x_2) &= I_0(x_2) + \sum_{k=1}^{\infty} I_k(x_1, x_2) \end{aligned} \tag{6.1}$$

where A_0, θ_0, and I_0 are slowly varying functions, and $A_k(x_1, x_2)$, $\theta_k(x_1, x_2)$, and $I_k(x_1, x_2)$ are quantities of the k-th order of smallness, rapidly oscillating with respect to x_1 and slowly varying with respect to x_2.

Differentiating by parts, from (5.4) and (5.3), we obtain

$$2\frac{dA}{dx_2} + 2\frac{\partial A}{\partial x_1} = 4\frac{\partial}{\partial x_1}(I\sqrt{\Phi}) - 4\frac{\partial}{\partial x_1}(\sqrt{\Phi}) + 4u(1 - I)\frac{1}{\sqrt{\Phi}}\frac{d\sqrt{\Phi}}{dx_2} - 4u\frac{d\sqrt{\Phi}}{dx_2} + 4\frac{dI}{dx_2}(u + \sqrt{\Phi}) \tag{6.2}$$

We expand all the functions containing $\Phi(A, \theta)$ in a Taylor series in the neighborhood of the point (A_0, θ_0)

$$\Phi = \Phi(A_0, \theta_0) + A_1 \frac{\partial \Phi(A_0, \theta_0)}{\partial A} + \theta_1 \frac{\partial \Phi(A_0, \theta_0)}{\partial x_1} + \ldots$$
$$\sqrt{\Phi} = \sqrt{\Phi(A_0, \theta_0)} + A_1 \frac{\partial \sqrt{\Phi(A_0, \theta_0)}}{\partial A} + \theta_1 \frac{\partial \sqrt{\Phi(A_0, \theta_0)}}{\partial x_1} + \ldots \tag{6.3}$$
$$\frac{1}{\sqrt{\Phi}} = \frac{1}{\sqrt{\Phi(A_0, \theta_0)}} + A_1 \frac{\partial}{\partial A} \frac{1}{\sqrt{\Phi(A_0, \theta_0)}} + \theta_1 \frac{\partial}{\partial x_1} \frac{1}{\sqrt{\Phi(A_0, \theta_0)}} + \ldots$$

Substituting (6.1) and (6.3) into Eq. (6.2) and discarding quantities of second order of smallness, we obtain

$$\frac{\partial A_1}{\partial x_1} + \frac{dA_0}{dx_2} = 2 \frac{\partial}{\partial x_1}(I_0 \sqrt{\Phi}) - 2 \frac{\partial}{\partial x_1}(\sqrt{\Phi}) + 2 \frac{\partial}{\partial x_1}(I_1 \sqrt{\Phi}) +$$
$$+ 2 \frac{\partial}{\partial x_1}\left[I_0 \left(A_1 \frac{\partial \sqrt{\Phi}}{\partial A} + \theta_1 \frac{\partial \sqrt{\Phi}}{\partial x_1}\right)\right] - 2 \frac{\partial}{\partial x_1}\left(A_1 \frac{\partial \sqrt{\Phi}}{\partial A} + \theta_1 \frac{\partial \sqrt{\Phi}}{\partial x_1}\right) +$$
$$+ \frac{2u(1 - I_0)}{\sqrt{\Phi}} \frac{d\sqrt{\Phi}}{dx_2} - 2u \frac{d\sqrt{\Phi}}{dx_2} + 2(u + \sqrt{\Phi}) \frac{dI_0}{dx_2} \tag{6.4}$$

Equating in (6.4) terms of the same order, we obtain

$$\frac{\partial}{\partial x_1}(I_0 \sqrt{\Phi}) - \frac{\partial}{\partial x_1} \sqrt{\Phi} = 0$$

Hence, $I_0 = 1$ and

$$\frac{\partial A_1}{\partial x_1} + \frac{dA_0}{dx_2} = 2 \frac{\partial}{\partial x_1}(I_1 \sqrt{\Phi}) - 2u \frac{d\sqrt{\Phi}}{dx_2} \tag{6.5}$$

Similarly to what has been done previously, from (5.5) and (5.3), we have

$$\frac{\partial \theta_1}{\partial x_1} = -\frac{I_1}{\sqrt{\Phi}} \frac{\partial \Phi}{\partial A} - \frac{1}{2} \frac{\partial^2 \Phi}{\partial A \partial x_1} \frac{d}{dx_2}(\Phi + 2u\sqrt{\Phi}) \frac{1}{2} \frac{\partial \Phi}{\partial A} \frac{d}{dx_2}\left(\frac{\partial \Phi}{\partial x_1}\right) \tag{6.6}$$

$$\frac{\partial I_1}{\partial x_1} = -u \frac{d}{dx_2} \frac{1}{\sqrt{\Phi}} = -u \left(\frac{dA_0}{dx_2} \frac{\partial}{\partial A} + \frac{d\theta_0}{dx_2} \frac{\partial}{\partial x_1}\right) \frac{1}{\sqrt{\Phi}} \tag{6.7}$$

Averaging (6.5) over the fast oscillations, we obtain

$$\frac{dA_0}{dx_2} = -2u \frac{dA_0}{dx_2} \frac{\partial}{\partial A} \langle \sqrt{\Phi} \rangle \tag{6.8}$$

Here, the brackets denote averaging over x_1:

$$\langle \sqrt{\Phi} \rangle = \frac{1}{\pi} \int_{(1-\sqrt{A})^2}^{(1+\sqrt{A})^2} \frac{\sqrt{\Phi}\, d\Phi}{2\sqrt{F(A_0, \Phi)}} = 1 + \frac{A_0}{2}$$

Thus, Eq. (6.8) gives

$$\frac{dA_0}{dx_2}(1 + u) = 0$$

Two cases are possible: $u = -1$ and $u \neq -1$.

7. We consider the case $u = -1$. This denotes that the longwave perturbation moves with velocity $u_2 = v_0$. Integrating Eq. (6.7), we obtain

$$I_1 = \frac{dA_0}{dx_2} \frac{\partial}{\partial A} \arcsin \frac{\sqrt{\Phi} - 1}{\sqrt{A_0}} + \frac{d\theta_0}{dx_2} \frac{1}{\sqrt{\Phi}} \tag{7.1}$$

From the condition $\langle n/n_0 \rangle = 1$, it follows that the constant of integration must be set equal to zero. Substituting (7.1) into (6.6) and averaging over the fast oscillations, we have

$$\frac{d\theta_0}{dx_2}\left\langle\frac{\partial^2\Phi}{\partial A\partial x_1}\frac{\partial\sqrt{\Phi}}{\partial x_1}-\frac{1}{\Phi}\frac{\partial\Phi}{\partial A}-1\right\rangle=\frac{dA_0}{dx_2}\left\langle\frac{2\partial\sqrt{\Phi}}{\partial A}\frac{\partial}{\partial A}\arcsin\frac{\sqrt{\Phi}-1}{\sqrt{A_0}}\right\rangle-\frac{dA_0}{dx_2}\left\langle\frac{\partial^2\Phi}{\partial A\partial x_1}\frac{\partial\sqrt{\Phi}}{\partial A}\right\rangle$$

Integrating by parts, we obtain

$$\frac{d\theta_0}{dx_2}=\frac{A_0}{\pi}\left(\frac{1}{\sqrt{A_0}}-\frac{A_0+1}{A_0\sqrt{A_0}}\ln 2\sqrt{A_0}\right) \tag{7.2}$$

Hence

$$\theta_0=\frac{2}{\pi}\left(\frac{2A_0+1}{\sqrt{A_0}}+\frac{1-A_0}{\sqrt{A_0}}\ln 2\sqrt{A_0}\right) \tag{7.3}$$

As is seen from (7.3), the longwave perturbation of the amplitude of the stationary wave leads to distortion of the phase of the latter.

8. We consider the case $u \neq -1$. Here, A_0 = const. Equation (6.7) implies

$$I_1=-\frac{u}{\sqrt{\Phi}}\frac{d\theta_0}{dx_2} \tag{8.1}$$

Substituting (8.1) into Eq. (6.6), and averaging over the fast oscillations, we obtain

$$\frac{d\theta_0}{dx_2}\left(-u\frac{\partial}{\partial A}\langle\ln\Phi\rangle-\frac{1}{2}\left\langle\frac{\partial^2\Phi}{\partial A\partial x_1}\frac{\partial}{\partial x_1}(\Phi+2u\sqrt{\Phi})\right\rangle+\frac{1}{2}\left\langle\frac{\partial\Phi}{\partial A}\frac{\partial^2\Phi}{\partial x_1^2}\right\rangle\right)=0$$

We can verify that the expression in parentheses does not equal zero. Therefore, θ_0 = const, with Eq. (8.1) giving $I_1 = 0$. Thus, if $u_2 \neq v_0$, the longwave perturbation of a stationary wave in the first approximation is absent. We show that in each approximation for $u_2 \neq v_0$, the perturbation is absent. We note that all quantities of the form

$$\frac{d}{dx_2}f(\Phi)=\left(\frac{dA_1}{dx_2}\frac{\partial}{\partial A}+\frac{d\theta_1}{dx_2}\frac{\partial}{\partial x_1}\right)f(\Phi)$$

are of second order of smallness.

Equations of the second approximation have the form

$$\frac{\partial A_2}{\partial x_1}+\frac{dA_1}{dx_2}=2\frac{\partial}{\partial x_1}(I_2\sqrt{\Phi})-2u\frac{d\sqrt{\Phi}}{dx_2}$$

$$\frac{\partial\theta_2}{\partial x_1}=-\frac{I_2}{\sqrt{\Phi}}\frac{\partial\Phi}{\partial A}-\frac{1}{2}\frac{\partial^2\Phi}{\partial A\partial x_1}\frac{d}{dx_2}(\Phi+2u\sqrt{\Phi})+\frac{1}{2}\frac{\partial\Phi}{\partial A}\frac{d}{dx_2}\left(\frac{\partial\Phi}{\partial x_1}\right)$$

$$\frac{\partial I_2}{\partial x_1}=-u\frac{d}{dx_2}\frac{1}{\sqrt{\Phi}}=-u\left(\frac{dA_1}{dx_2}\frac{\partial}{\partial A}+\frac{d\theta_1}{dx_2}\frac{\partial}{\partial x_1}\right)\frac{1}{\sqrt{\Phi}}$$

This system of equations coincides with (6.5)-(6.7), if in it we replace A_2, θ_2, A_1, θ_1, and I_2, by, respectively, A_1, θ_1, A_0, θ_0, and I_1. Similarly to the preceding, we obtain

$$A_1=\theta_1=I_1=0,\quad A_2=A_2(x_2),\quad \theta_2=\theta_2(x_2)$$

By induction it is easy to show that all the quantities of the series (6.1), beginning with the first and higher, equal zero. Thus, the stationary wave of the envelope can move only with velocity of the unperturbed stream.

9. The results obtained can be understood by analyzing the law of dispersion of waves of the space charge

$$k^2/\omega^2=\varepsilon(\omega)/v_0^2 \tag{9.1}$$

where

$$\varepsilon(\omega)=(1\pm\omega_p/\omega)^2$$

or

$$\omega/k=v_0\pm\omega_p/k \tag{9.2}$$

We note that the analysis usually used for the expansion of the function $\varepsilon(\omega)$ in powers of ω is not used here, since for $\omega = 0$, there is a singularity. It is shown below that Eqs. (9.1) and (9.2) are simultaneously also a nonlinear law of dispersion. It is characteristic that the wave amplitude does not appear in the dispersion law. For $\omega \to \infty$ or $k \to \infty$ the dispersion vanishes and the phase velocity approaches the velocity of the unperturbed stream v_0.

We show that the character of the envelope waves in an electron stream is due to the final characteristics of the dispersion relations (9.1), (9.2). It is not difficult to verify that the initial system (3.1) can be found from a variational principle, in which the density of the Lagrangian function

$$L = \frac{1}{8\pi}\left(\frac{\partial \varphi}{\partial x}\right)^2 - mn\left[\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2\right] + e\varphi(n - n_0) + mnC(t) \tag{9.3}$$

where

$$\psi(x, t) = \int_x^{x_0} v(x', t)\, dx'$$

$$C(t) = \frac{1}{2} v^2(x_0, t) - \frac{e}{m}\varphi(x_0, t)$$

The generalized coordinates are the field variables $n(x, t)$, $\psi(x, t)$, and $\varphi(x, t)$. As was shown, the system (3.1) admits a stationary solution, in which all the quantities are functions of the combinations $kx - \omega t$. For perturbations having wavelengths much greater than the wavelength of the stationary wave, the amplitude A, the wave number k, and the frequency ω are slowly varying functions of the time and the coordinate. We can therefore assume that a locally stationary solution holds at each point; however, the amplitude, the wavelength, and the frequency vary from point to point. In [12], it is shown that averaged equations for the slowly varying functions A, k, and ω are obtained from the averaged Lagrangian by variation with respect to A, k, and ω.

Substituting the stationary solution into the Lagrangian and averaging over the period of the stationary wave, we have

$$\langle L \rangle = \frac{1}{8\pi}\left\langle\left(\frac{\partial \varphi}{\partial x}\right)^2\right\rangle - en_0\langle \varphi \rangle \tag{9.4}$$

In (9.4), we take into account that variation with respect to n gives the equation of motion

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 = \frac{e}{m}\varphi + C(t) \tag{9.5}$$

Substitution of the stationary solution into it transforms (9.5) into an identity, and causes the expression

$$\left\langle mn\left[\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2\right] - e\varphi n - mnC(t)\right\rangle$$

to vanish.

For the stationary wave

$$\varphi = \frac{(kv_0 - \omega)^2 m}{2ek^2}\left\{\Phi\left[A, \frac{\omega_p}{|kv_0 - \omega|}(kx - \omega t) + \theta\right] + 1\right\}$$

because

$$x_1 = \omega_p \frac{x - ut}{|v_0 - u_1|} = \frac{\omega_p}{|kv_0 - \omega|}(kx - \omega t)$$

The function $\Phi = \Phi(A, x_1 + \theta)$ is defined by Eq. (4.3). Its properties are described, for example, in [8]. Since the period Φ equals 2π for all values of the amplitude $A(0 < A < 1)$, we obtain

$$|kv_0 - \omega| = \omega_p$$

i.e.,

$$\varphi = \frac{2\pi e n_0}{k^2}\{\Phi(A, kx - \omega t + \theta) + 1\} \tag{9.6}$$

Thus, the linear dispersion equation is simultaneously also nonlinear. These are the specific properties of the problem under consideration.

Substituting (9.6) into (9.4) and averaging, we obtain

$$\langle L \rangle = -\frac{2\pi e^2 n_0^2}{k^2} A$$

Using the method of [12], we obtain the Euler equation for the averaged Langrangian L

$$\frac{\partial \langle L \rangle}{\partial A} = 0, \qquad \frac{\partial}{\partial x}\frac{\partial \langle L \rangle}{\partial k} = 0$$

or

$$\frac{1}{k^2} - 2\frac{A}{k^3}\frac{\partial k}{\partial A} = 0, \qquad \frac{\partial}{\partial x}\left(\frac{1}{k^2}\frac{\partial A}{\partial k} - 2\frac{A}{k^3}\right) = 0$$

Hence, it follows that

$$A / k^2 = \text{const} \tag{9.7}$$

Using the dispersion equation $\omega = kv_0 \pm \omega_p$, and the equation

$$\partial k / \partial t = -\partial \omega / \partial x$$

we obtain

$$\partial k / \partial t + v_0 \partial k / \partial x = 0$$

i.e.,

$$k = k(x - v_0 t)$$

Equation (9.7) yields

$$A = A(x - v_0 t)$$

which was to be proved.

We note that in the literature there is a known exact solution of the system of equations (3.1) in Lagrangian variables (see, e.g., [13, 14]). However, the use of this solution in the present study complicates the difficulty of transforming in explicit form from Lagrangian variables to Euler variables.

Thus, the nonlinear distortion of envelope waves is evidently possible only in the multivelocity streams.

The results obtained naturally refer not only to electron streams, but also to waves in a cold plasma.

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